MATH 3235 Probability Theory

11/15/22

MidTerm Review.

Ex. 1.

$$\mathbb{E}(X^k) = \int_{-\infty}^{\infty} x^k e^{-\frac{x^2}{2}} dx$$

$$x e^{-\frac{x^2}{2}} = -\frac{d}{dx} e^{-\frac{x^2}{2}}$$

$$\int_{-\infty}^{\infty} x^{k-1} x e^{-\frac{x^2}{2}} dx = -x^{k-1} e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} +$$

$$(k-1) \int_{-\infty}^{\infty} x^{k-2} e^{-\frac{x^2}{2}} dx =$$

$$= (k-1)\, \mathbb{E}(X^{k-2})$$

$$m_k = (k-1)\, m_{k-2}$$

$$m_{2k} = \prod_{n=1}^{k} (2n-1) = (2k-1)!! = \frac{(2k)!}{2^n k!}$$

k odd $\quad m_k = 0$

$$M_X(t) = e^{\frac{t^2}{2}} = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n n!}$$

K-th derivative is

k! coeff of t^k

$$m_{2k} = (2k)! \frac{1}{2^k k!}$$

$$\int x^k e^{-\frac{x^2}{2}} dx =$$

$$\frac{d}{dx} \frac{x^{k+1}}{k+1} = x^k$$

$$- \int \frac{x^{k+1}}{k+1} \left(-x e^{-\frac{x^2}{2}}\right) dx =$$

$$\frac{1}{k+1} \int x^{k+2} e^{-\frac{x^2}{2}} dx = \frac{m_{k+2}}{k+1}$$

$$m_0 = 1 .$$

Ex. 2: Black - Scholes

$T+1 \qquad T$

$$X_{t+1} = \Delta_t X_t \qquad X_0 < 1$$

$$X_t = \prod_{s=0}^{t-1} \Delta_t$$

$$\Delta_t = e^{Z_t} \quad \text{with} \quad Z_t \sim \mathcal{N}(\mu, \sigma^2)$$

$$\left.\begin{aligned} X_{t+dt} &= \Delta_{dt} X_t \\ X_{t+dt} &= (1 + dt\, Z_t) X_t \\ \frac{X_{t+dt} - X_t}{dt} &= Z_t X_t \end{aligned}\right]$$

$$f_\Delta(d) \qquad \Delta_t = e^{Z_t} \quad \text{with}$$

$$Z_t \sim \mathcal{N}(\mu, \sigma^2)$$

$$\frac{1}{\delta\sqrt{2\pi\sigma^2}} e^{-\left(\frac{(\log\delta-\mu)^2}{2\sigma^2}\right)}$$

$$\frac{dz}{d\delta}$$

$$z = \log\delta \Rightarrow \delta = e^{z}$$

$$\Delta_t = e^{z_t} \qquad z_t \sim \mathcal{N}(\mu, \sigma^2)$$

$$X_t = e^{T_t} \qquad T_t = \sum_{s=0}^{t-1} z_t \sim \mathcal{N}(t\mu, t\sigma^2)$$

$$f_{X_t}(x) = \frac{1}{x\sqrt{2\pi t\sigma^2}} \exp\left(-\frac{(\log x - t\mu)^2}{2t\sigma^2}\right)$$

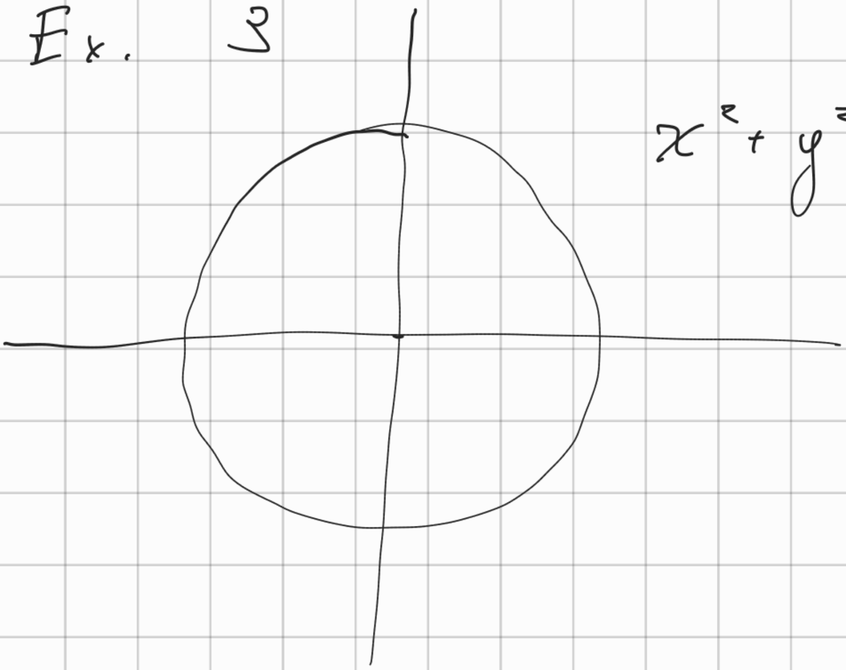
$$\mathbb{P}(X_t > \bar{x}) = \mathbb{P}(T_{10} > \log\bar{x}) =$$

$$\mathbb{P}\left(\frac{T_{10} - 10\mu}{10\sigma^2} > \frac{\log\bar{x} - 10\mu}{10\sigma^2}\right) =$$

$$1 - \Phi\left(\frac{\log\bar{x} - 1}{2}\right) = 0.75$$

$$\log \bar{x} = 1 - \Phi_{+}^{-1}(0.25) \cdot 2$$

Ex. 3



$x^2 + y^2 \leq 1$

(U, V) are $N(0,1)$ independent

$$f_{UV} = \frac{1}{2\pi} e^{-\frac{u^2+v^2}{2}}$$

$$R = \sqrt{U^2 + V^2}$$

$$U = R \cos \theta$$

$$V = R \sin \theta$$

$$f_{\theta, R} = \frac{1}{2\pi} r e^{-\frac{r^2}{2}} \qquad 0 \leq \theta \leq 2\pi$$

$$r > 0$$

R Θ are independent

$$f_R(r) = re^{-\frac{r^2}{2}} \Rightarrow$$

$$F_R(r) = 1 - e^{\frac{r^2}{2}}$$ easy to invert

Z uniform in $0,1$

$$R = \sqrt{-2 \log Z}$$

given Θ I must compute

$\cos\Theta$ $\sin\Theta$

(X, Y) are uniform in

The unit circle $\Rightarrow$

$$\left(\frac{X}{\sqrt{X^2+Y^2}}, \frac{Y}{\sqrt{X^2+Y^2}}\right) = (\cos\Theta, \sin\Theta)$$

Θ uniform in $[0, 2\pi]$

4). X_i i.i.d. r.v $X \sim Exp(1)$

$$T_n = \sum_{i=1}^{n} X_i$$

(a) p.d.f. of T_n.

$$f(x_1 \dots x_n) = e^{-\sum_{i=1}^{n} x_i}$$

$t_1 = x_1$

$t_2 = x_1 + x_2$

$t_3 = x_1 + x_2 + x_3$

$\vdots$

$t_n = x_1 \dots x_n$

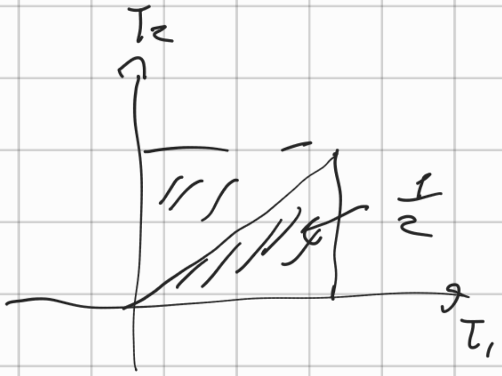
$$e^{-t_n}$$

$$J = \begin{pmatrix} 1 & & & 0 \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$\det J = 1$

$$f_{T_1,n}(\tau_1 \dots \tau_n) = e^{-\tau_n} \qquad 0<\tau_1<\tau_2<\dots<t_n$$

$$f_{T_n}(\tau_n) = \int_{0<t_1<t_2 \dots <t_n} dt_1 \dots dt_{n-1}\, e^{-t_n} = \frac{\tau_n^{n-1}}{(n-1)!} e^{-t_n}$$



$$\mathbb{P}(N_t < n) = \mathbb{P}(T_n > t)$$

$$\mathbb{P}(N_t \le n) = \int_t^\infty \frac{s^n}{n!} e^{-s}\, ds =$$

$$= \int_0^\infty \frac{(s+t)^n}{n!} e^{-(s+t)} =$$

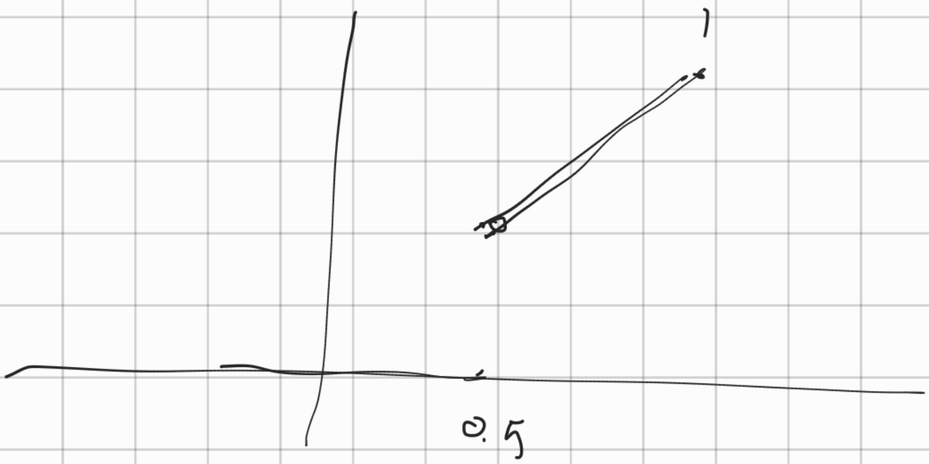
$$e^{-t} \sum_{k=0}^{n} \frac{t^k}{n!} \binom{n}{k} \overbrace{\int_0^\infty s^{n-k} e^{-s}\, ds}^{(n-k)!} =$$

$$= e^{-t} \sum_{k=0}^{n} \frac{t^k}{k!}$$

$$P(N_t = n) = \frac{t^n}{n!} e^{-t}$$

$t_1 \quad t_2$

$$\left(N_{t_2} - N_{t_1}\right) \simeq N_{t_2 - t_1}$$



$$f_Z(z) = \sum_y f_X(z-y)\, p_Y(y)$$